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Exam I: Discrete Math, MTH 213, Fall 2017

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Score = $\overline{70}$

~~70~~ 70 / 70

QUESTION 1. (3 points) Make a brief argument in order to convince me that $|Q \cap (0, 0.002)| = |N^*|$

we know that $|Q| = \infty$ and Q is countable $\Rightarrow |Q| = |N^*|$

we also know that $|(0, 0.002)| = \infty$ but it is uncountable

$\Rightarrow Q \cap (0, 0.002)$ is countable because countable $\cap$ uncountable = countable

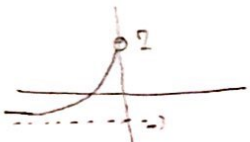
Thus, $|Q \cap (0, 0.002)| = \infty$ and it is countable

$\Rightarrow |Q \cap (0, 0.002)| = |N^*|$

3/3

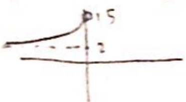
QUESTION 2. (4 points) Construct bijective functions in order to convince me that $|(-3, 7)| = |(2, 15)|$ make a rough sketch of each graph in order to show bijection

① $f : (-\infty, 0) \longrightarrow (-3, 7)$



$f(x) = 10e^x - 3$, it is a bijective function $\Rightarrow |(-\infty, 0)| = |(-3, 7)| = \infty$

② $F : (-\infty, 0) \rightarrow (2, 15)$



$f(x) = 13e^x + 2$, it is a bijective function $\Rightarrow |(-\infty, 0)| = |(2, 15)| = \infty$

from ① and ②

$|(-3, 7)| = |(-\infty, 0)| = |(2, 15)| = \infty$

4/4

QUESTION 3. (4 points) Find $(238)_9 \cdot (17)_9$.

$$
\begin{array}{r}
(238)_9 \\
\times\ (17)_9 \\
\hline
1802 \\
+\ 2380 \\
\hline
\boxed{4282}
\end{array}
\Rightarrow (4282)_9
$$

Find $(207)_8 - (66)_8$

$$
\begin{array}{r}
207 \\
-\ 66 \\
\hline
121
\end{array}
\Rightarrow (121)_8
$$

QUESTION 4. (4 points) Let $a = 424$, $b = 159$. Use the method we discussed in class to find $gcd(a, b)$.

$159 \overline{)424}$ → 2; $424 - 318 = 106$ $\Rightarrow$ $106 \overline{)159}$ → 1; $159 - 106 = 53$ $\Rightarrow$ $53 \overline{)106}$ → 2; $106 - 106 = 0$

$\Rightarrow gcd(424, 159) = 53$

4/4

Find $LCM(a, b)$.

$$LCM(424, 159) = \frac{424 \times 159}{gcd(424, 159)} = \frac{424 \times 159}{53} = 1272$$

3152   135   54   81

QUESTION 5. (5 points) Use math induction to prove that $18 \mid (5^{6n} - 1)$ for every positive integer $n \geq 1$

① prove it is true for $n=1 \Rightarrow 18 \mid 15624$ True $\frac{15624}{18} = 868$

② Assume it is true for $n=k$, for $k \geq 1 \Rightarrow 18 \mid (5^{6k} - 1)$

③ Prove it is true for $n=k+1 \Rightarrow$

5/5

$18 \mid (5^{6k+6} - 1) \Rightarrow 18 \mid (5^{6k} \cdot 5^{6}) - 1 \Rightarrow 18 \mid 5^6(5^{6k}-1) + \underbrace{15624}_{(5^6-1)}$

$\Rightarrow 18 \mid \underbrace{5^6(5^{6k}-1)}_{18 \mid \text{ by } ②} + \underbrace{15624}_{18 \mid \text{ by } ①} \Rightarrow 18 \mid (5^{6(k+1)} - 1)$

QUESTION 6. (3 points) Use direct prove (max. 3 line) to show that for every positive integer $n \geq 1$. we have

$$3^n = nC0 \cdot 2^n + nC1 \cdot 2^{n-1} + nC2 \cdot 2^{n-2} + \ldots + nC(n-1) \cdot 2 + 1$$

we know that $(x+1)^n = nC0\, x^n + nC1\, x^{n-1} + nC2\, x^{n-2} \ldots nC\ldots$

if we replace x by 2 we will get:

$3^n = nC0\, 2^n + nC1\, 2^{n-1} + nC2\, 2^{n-2} \ldots\ldots nC(n-1)\, 2 + 1$

3/3

QUESTION 7. (6 points) Let $x \in R$ and assume that $\sqrt{x}$ is irrational. Prove that $\sqrt[4]{x}$ is irrational. [use Contradiction, 2 to 3 lines proof]

we assume that $\sqrt[4]{x}$ is rational $\Rightarrow$

$\sqrt[4]{x} = \frac{a}{b}$, $a, b \in \mathbb{Z}$ and $b \neq 0$ and $\gcd(a,b) = 1$, if we square both sides

$\underbrace{\sqrt{x}}_{\text{irrational}} = \underbrace{\frac{a^2}{b^2}}_{\text{rational}}$ 3/3 $\Rightarrow$ contradiction $\Rightarrow \sqrt[4]{x}$ is irrational

Convince me that $\sqrt{60}$ is irrational (Hint: First write $\sqrt{60} = 2 \times \sqrt{15}$. So you only need to show $\sqrt{15}$ is irrational).

we know that $\sqrt{60} = 2\sqrt{15}$, Assume $\sqrt{15}$ is rational

$\Rightarrow \sqrt{15} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

$15 = \frac{a^2}{b^2}$, we know that a, b are odd $\Rightarrow 15 = \frac{(2k+1)^2}{(2m+1)^2}$

$\Rightarrow 15 \times 4m^2 + 15 \times 4m + 15 = 4k^2 + 4k + 1 \Rightarrow$

$60m^2 + 60m + 14 = \underbrace{4(k^2+k)}$

4/4 $4 \mid 4(k^2+k)$ then 4 should $4 \mid 60m^2 + 60m + 14$

but $4 \nmid 14 \Rightarrow$ contradiction $\Rightarrow$ ...

$\sqrt{15}$ is irrational, and we know that

irrational $\times$ non-zero rational = irrational

$\Rightarrow$ $2\sqrt{15} = \sqrt{60}$ is irrational

M. Said Alghabra

QUESTION 8. Write T or F ONLY(4 points) Let $A = \{x, \{x\}, 3, \{y\}, \{3, x\}, 4\}$

$\{y\} \subset A \Rightarrow y \in A$

(i) $\{x\} \subset P(A)$ F

(ii) $\{4, x\} \subset P(A)$ F

(iii) $\{3, x\} \in P(A)$. T

(iv) $\{\{x\}, \{y\}, \{3\}\} \subset P(A)$ F

$\{x\} \in PA \Rightarrow x \in A$

$\{y\} \in PA \Rightarrow y \in A$

$\{3\} \in PA \Rightarrow 3 \in A$

QUESTION 9. Write T or F ONLY(6 points)

(i) $\{(1, 1), (a, a), (-2, -2)\}$ is an equivalence relation on $A = \{1, a, -2\}$ T

(ii) $|(0,\ 0.001)| = |Z|$ (note $(0,\ 0.001)$ is the set of all real numbers between 0 and 0.001) F

(iii) $\{(2, 2), (3, 3), (1, 1), (1, 3)\}$ is a partial order relation on $A = \{1, 2, 3\}$ T

(iv) $\{(a, a), (b, b), (c, c), (a, c), (c, b)\}$ is a partial order relation on $A = \{a, b, c\}$ F

(v) $\forall x \in N^* \exists y \in R$ such that $y^x < 0$ F

(vi) $\exists x \in N^*$ such that $y^x \geq 0 \ \forall y \in R$ T

QUESTION 10. (5 points) Solve over Z. $3x \equiv 6 \ (mod\ 9)$

$3x \equiv 6 \pmod 9$

$\gcd(3,9) = 3$ and $3 \mid 6 \Rightarrow$ we have 3 solutions on Z_6

$\Rightarrow$ solution set $= \{2, 5, 8\}$ over Z_9

over $Z \Rightarrow$ solution set $= \{2+9n, 5+9m, 8+9k\}$ where $n, m, k \in Z$

1
9 10
18 19 24
27 28
36 37
45 46
54 55
63 64
72 73

QUESTION 11. (8 points) Let x be the number of all females in an event. Given x is an even number (*i.e.*, $x \equiv 0\ (mod\ 2)$). If x is divided by 11, the remainder is 5. If x is divided by 7, the remainder is 6. Use the Chinese remainder Theorem to answer the following:

a) If $0 < x < 154$, what is the value of x?

1- $x \equiv 0 \pmod 2$

2- $x \equiv 5 \pmod{11}$

3- $x \equiv 6 \pmod 7$

$\gcd(m_i, m_j) = 1$ for every i, j

$m = 154$, $Q_1 = \frac{m}{m_1} = 77$, $Q_2 = \frac{m}{m_2} = 14$

$Q_3 = \frac{m}{m_3} = 22$

$y_1 = [77 \bmod 2]^{-1} = (1)^{-1} = 1$

$y_2 = [14 \bmod 11]^{-1} = (3)^{-1} = 4$

$y_3 = [22 \bmod 7]^{-1} = (1)^{-1} = 1$

$\Rightarrow x = [0 \times 1 \times 77 + 5 \times 14 \times 4 + 6 \times 22 \times 1] \bmod 154$

$\Rightarrow \boxed{x = 104}$

over Z, $x = 104 + 154n$, $n \in Z$

b) If $154 < x < 308$, what is the value of x?

let $n = 1 \Rightarrow \boxed{x = 258}$

40/40

$a \leq b$ $b \subseteq a$

QUESTION 12. (8 points) Let $A = \{0, 4, 5\}$. Define a relation, say $\leq$, on $P(A)$ such that $\forall a, b \in P(A)$, we have $a \leq b$ iff $b \subseteq a$.

1. Convince me that $\leq$ is a partial order relation on $P(A)$.

$a \leq b$
$b \subseteq a$

1- Symmetric, $\forall a \in P(A)$, $a \subseteq a$ (because every set is a subset or equal to itself) $\Rightarrow a \leq a$ ✓

2- Anti-Reflexive, for $a \neq b$, $a, b \in P(A)$ if $a \leq b \Rightarrow b \subseteq a$ and we know that $P(A)$ has all combinations of subsets of A each of them only once $\Rightarrow$ if $b \subseteq a$ and $a \subseteq b$ then for sure $a = b$ and that contradicts our first condition $\Rightarrow b \nleq a \Rightarrow a \nleq a$

3- Transitive, if $a \leq b$ and $b \leq c$ then we know $b \subseteq a$ and $c \subseteq b$ then clearly $c \subseteq a$ $\Rightarrow a \leq c$ ✓

$\{0,4\}$ $\{0,5\}$ a

$a \leq \{0,4\}$

$\{4\} \subseteq \{0,4\}$

a $a \leq \{0\}$

$\{5,4\}$ $\{0\}$

$\{4,5\}$ $\{5\}$

2. Find $\{0, 4\} \wedge \{0, 5\}$ $= \{0,4,5\}$ ✓
3. Find $\{0, 4\} \wedge \{4\}$ $= \{0,4\}$ ✓
4. Find $\{5, 4\} \vee \{0\}$ $= \emptyset$ ✓
5. Find $\{4, 5\} \vee \{5\}$ $= \{5\}$ ✓
6. If possible, find the Least element and the greatest element of $\leq$.

greatest is $\emptyset$ ✓

Least is $\{0,4,5\}$ ✓

QUESTION 13. (10 points) Let $A = \{1, 2, 8, 10, 11, 19, 22\}$, $B = \{0, 1, 2, 3, -1, -2, -3\}$. Define $=$ on A such that $\forall a, b \in A$, we have $a = b$ iff $a - b \in B$.

1) Convince me that $=$ is an equivalence relation on A.

① Symmetric: $\forall a \in A$, $a - a = 0 \in B \Rightarrow a = a$ ✓

② Reflexive: if $a = b \Rightarrow a - b \in B$ let $a - b = C \in B$ $\Rightarrow b - a = -(a-b) = -C$ and we see that all the numbers in B has there positive and negative value $\Rightarrow b - a = -C \in B$ $\Rightarrow b = a$

③ Transitive: if $a = b$ and $b = c$ $\Rightarrow a - b \in B$ and $b - c \in B$

we see this case only with 8, 10, 11

So $8 - 10 = -2 \in B$ and $10 - 11 = -1 \in B$

$\Rightarrow 8 - 11 = -3 \in B \Rightarrow 8 = 11$ and $a = c$ ✓ OK

2) Find all equivalence classes

$[1] = \{1, 2\}$ ✓

$[8] = \{8, 10, 11\}$ ✓

$[19] = \{19, 22\}$ ✓

3) Find all elements of the relation $=$.

$\{(1,1), (2,2), (8,8), (10,10), (11,11), (19,19), (22,22), (1,2), (2,1), (8,10), (10,8), (8,11), (11,8), (10,11), (11,10), (19,22), (22,19)\}$ ✓

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